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A time dependent solution for an ionization chamber space charge density.

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Introduction

This document presents a time dependent solution of the continuity equation. This equation describes the space charge density inside an ionization chamber subjected in a continuous external ionization density rate N. The result is applied for the pulsed beam case and a formula is derived for saturation free chamber operation. This formula takes into account the time duration of the pulsed beam.

The continuity equation.

Consider a parallel plate ionization chamber with gap distance d and electrode area A, subjected to an ionization rate density N. Assume that the chamber electrode at x=0 is held at ground potential and the electrode at x=d at $-V_0$. For the potential and electric field inside the chamber we have:

$$\phi(x,t=0) = -\frac{V_0}{d}x$$

$$\vec{E}(x,t) = -\vec{\nabla}\phi(x,t) = -\frac{\partial\phi}{\partial x}(x,t)\overrightarrow{u_x}$$

$$\vec{E}(x,t=0) = \frac{V_0}{d}\overrightarrow{u_x}$$

$$\vec{\nabla}\vec{E}(x,t) = \frac{\partial E}{\partial x}(x,t) = 4\pi\rho(x,t)$$

According to this model the electrons are moving towards the ground electrode and the positive ions toward x=d.

The number of positive ions between x and x+dx, inside the active region of the chamber, at time t+dt is given by:

$$\rho(x,t+dt)Adx = \rho(x,t)Adx + NAdxdt + \rho(x,t)\upsilon(x,t)Adt - \rho(x+dx,t)\upsilon(x+dx,t)Adt \Rightarrow$$

$$\frac{\partial\rho}{\partial t}(x,t) = N - \frac{\partial}{\partial x}\left(\rho(x,t)\upsilon(x,t)\right)$$

The continuity equation essentially states that the ion density rate of change at time t and point x, is equal to the external ionization rate minus the charge lost at point x+dx (drifting towards x=d), plus the charge gained at point x (drifting towards x=d). The ion drift velocity (absolute value) is given by:

$$\upsilon(x,t) = \mu E(x,t)$$

If we substitute into the continuity equation we get:

$$\frac{\partial\rho}{\partial t}(x,t) = N - \mu\frac{\partial}{\partial x}\left(\rho(x,t)E(x,t)\right) = N - \mu E(x,t)\frac{\partial\rho}{\partial x}(x,t)) - \mu\rho(x,t)\frac{\partial E}{\partial x} \Rightarrow$$

$$\frac{\partial\rho}{\partial t}(x,t) = N - 4\pi\mu\rho^2(x,t) - \mu E(x,t)\frac{\partial\rho}{\partial x}(x,t)$$

A time dependent solution of the continuity equation.

We will assume that the space charge inside the chamber consists entirely of slow moving positive ions and we will seek time dependent solutions of the continuity equation with the constraint:

$$\rho(x, t=0) = 0$$

We will seek solutions of the above equation which are independent of x but they only depend on time t. This is true at least for some time at the beginning of the external ionization, as predicted by the short pulse approximation. [Ref. 1] the continuity equation takes the form:

$$\frac{d\rho}{dt}(t) = N - 4\pi\mu\rho^2(t) \Rightarrow \frac{d\rho}{\rho^2 - \frac{N}{4\pi\mu}} = -4\pi\mu dt \Rightarrow$$

$$\frac{1}{\sqrt{\frac{N}{\pi\mu}}}\int_0^{\rho} d\rho\left(\frac{1}{\rho - \sqrt{\frac{N}{4\pi\mu}}} - \frac{1}{\rho + \sqrt{\frac{N}{4\pi\mu}}}\right) = -4\pi\mu\int_0^{t} dt \Rightarrow$$

$$\ln\left(\frac{|\,\rho - \sqrt{\frac{N}{4\pi\mu}}\,|}{\rho + \sqrt{\frac{N}{4\pi\mu}}}\right) = -4t\sqrt{N\pi\mu} \Rightarrow$$

$$\rho(t) = \sqrt{\frac{N}{4\pi\mu}}\left(\frac{1 - e^{-4t\sqrt{N\pi\mu}}}{1 + e^{-4t\sqrt{N\pi\mu}}}\right) \Rightarrow$$

$$\rho(t) = \sqrt{\frac{N}{4\pi\mu}}\tanh\left(t\sqrt{4\pi\mu N}\right)$$

The last equation shows the time development of the space charge inside the chamber. It satisfies the initial condition of 0 space charge at t=0 (when the ionization begins) and it has an asymptotic value of:

$$\rho(t \longrightarrow \infty) = \sqrt{\frac{N}{4\pi\mu}}$$

The electric field can be easily calculated. From the short pulse approximation results [Ref.1] replace the charge density with the time dependent one we have found.
The saturation free condition is:

$$\frac{V}{d^2} \geq 4\pi\rho(t) = \sqrt{\frac{4\pi N}{\mu}} \tanh\left(t\sqrt{4\pi\mu N}\right)$$

Suppose the duration of a pulsed beam is T_0 and the total ionization density produced (by neglecting the ion movement) is ρ_0. Then, the corresponding ionization density rate N lasting for as long as the beam pulse lasts, is:

$$N = \frac{\rho_0}{T_0}, 0 \leq t \leq T_0, N = 0 \quad otherwise$$

The saturation free condition takes the form:

$$\frac{V}{d^2} \geq \sqrt{\frac{4\pi\rho_0}{\mu T_0}} \tanh\left(t\sqrt{\frac{4\pi\mu\rho_0}{T_0}}\right), 0 \leq t \leq T_0$$

At time t=T_0 the maximum space charge inside the chamber has been reached, and the saturation free condition takes its most stringent form.

$$\frac{V}{d^2} \geq \sqrt{\frac{4\pi\rho_0}{\mu T_0}} \tanh\left(\sqrt{4\pi\mu\rho_0 T_0}\right)$$

The above equation defines the saturation free operation of the chamber for an arbitrary duration pulsed beam. It also defines a meaning of the term **short** pulse approximation. That is, if:

$$4\pi\mu\rho_0 T_0 << 1$$

By using:

$$\tanh(x) \approx x - \frac{x^3}{3}$$

We get:

$$\frac{V}{d^2} \geq 4\pi\rho_0 \left(1 - \frac{4\pi\mu\rho_0 T_0}{3}\right)$$

The first term is the short pulse approximation, while the second one is the first order correction with respect to the time duration of the beam.
In the case of the very long, continuous and constant ionization density rates N, tha saturation free equation takes the form:

$$\frac{V}{d^2} \geq \sqrt{\frac{4\pi N}{\mu}} \tanh\left(t\sqrt{4\pi\mu N}\right) \xrightarrow{t \longrightarrow \infty} \sqrt{\frac{4\pi N}{\mu}} \Rightarrow$$

$$\frac{V}{d^2} \geq \sqrt{\frac{4\pi N}{\mu}}$$

The last result differs from the prediction of other models [Ref. 1] by a factor of 2.

Epilogue

An exact solution of the time dependent continuity equation, describing the space charge density inside an ionization chamber under an external ionization rate N, has been derived. Although we made the arbitrary assumption that the space charge inside the chamber is space independent, the derived exact solution satisfies our initial condition (zero space charge density at the onset of the external ionization cause) and for short beam pulses it agrees with the short pulse approximation results. Recent experiments at Brookhaven National Laboratory have demonstrated the validity of the short pulse approximation model. For a continuous, externally caused, ionization density rate N, the saturation free condition result differs from the one predicted by the steady state model by a factor of 2. That model assumes space charge density independent of time and dependent only on space x. No analytical solution of the continuity equation has ever been found, that, while it satisfies the initial condition no space charge at t=0, it yields the steady state model result at very long times.
The value (or time range validity) of the model and result presented in this document can only be assessed by experiments where the beam pulses are long as defined by the condition:

$$4\pi\mu\rho_0 T_0 \geq 1$$

Another interesting prediction of the model is that the concept of long or short pulses duration is not depending on the chamber parameters (gap distance or voltage) but on a time parameter:

$$\tau \equiv \frac{1}{4\pi\mu\rho_0}$$

However a serious drawback of the model is that it does not satisfy the conservation of charge. Therefore it cannot considered to be a physical solution of a real chamber.
In order to see this, we express the charge conservation as:

$$\int_0^x \frac{\partial p}{\partial t} dx = Nx - \mu E(x,t) \frac{\partial p}{\partial x}$$

By direct substitution of the p(x,t) expression, we do not find the value of E(x,t) satisfying the potential boundary conditions.

For the MINOS muon chambers operating with He gas:

$$\mu \approx 11.0 cm^2 \sec^{-1} Volts^{-1}$$

$$\rho_0 \approx 10^8 x 1.6 x 10^{-19} \frac{Cb}{cm^2} x 20 cm^{-1} \approx 3.2 x 10^{-10} \frac{Cb}{cm^3}$$

$$T_0 \approx 8.0 \mu \sec$$

$$\tau \equiv \frac{1}{4\pi\mu\rho_0} \approx 25.12 \mu \sec$$

$$4\pi\mu\rho_0 T_0 \approx 0.31847$$

And the correction one should apply to the space charge density predicted by the short pulse approximation, would be only 0.905.

References

1.Principles of ionization chamber operation under intense ionization rates. C.Velissaris, NUMI-717
2.Nuclear Radiation Detectors. J.Sharpe, John Wiley and Sons.
3.Space charge in ionization detectors and the NA48 electromagnetic calorimeter. S. Palestini et. al. NIM A421 (1999) 75-89